Formalism for the Subhalo Mass Function in the Tidal-limit Approximation

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ABSTRACT

We present a theoretical formalism by which the global and the local mass functions of dark matter substructures (dark subhalos) can be analytically estimated. The *global* subhalo mass function is defined to give the total number density of dark subhalos in the universe as a function of mass, while the *local* subhalo mass function counts only those subhalos included in one individual host halo. We develop our formalism by modifying the Press-Schechter theory to incorporate the followings: (i) the internal structure of dark halos; (ii) the correlations between the halos and the subhalos; (iii) the subhalo mass-loss effect driven by the tidal forces. We find that the resulting (cumulative) subhalo mass function is close to a power law with the slope of ~ -1, that the subhalos contribute approximately 10% of the total mass, and that the tidal stripping effect changes the subhalo mass function self-similarly, all consistent with recent numerical detections.

Subject headings: cosmology:theory — large-scale structure of universe

1. INTRODUCTION

The dark halo substructures (*dark subhalos*) are the dynamically distinct, self-bound objects in virialized dark matter halos. The presence of substructures in the dark matter halos is a generic picture of the cold dark matter (CDM) cosmology. Recent numerical simulations of ultra-high resolution indeed confirmed that the dark halos are not smooth structureless objects but clumpy systems marked by a wealth of substructures (Tormen *et al.* 1998; Klypin et al. 1999; Okamoto & Habe 1999; Ghigna et al. 2000; Springel et al. 2002; Zhang et al. 2002; De Lucia et al. 2003; Hayashi et al. 2003; Zenter & Bullock 2003).

Recently, the mass function of dark subhalos has drawn sharp attentions (Fujita et al. 2002; Sheth 2003; Blanton 2003) especially because of its connection to the galaxy luminosity

function. Yet, it is not an easy task to derive the subhalo mass function either in numerical or analytical ways. The numerical approach to the subhalo mass function using N-body simulations still suffers from resolution effects related to the so called over-merging problem (Klypin et al. 1999). Even recently available ultra-high resolution simulations are capable of producing only the local subhalo mass function, i.e., the mass function of the subhalos within one individual dark halo (Okamoto & Habe 1999; Ghigna et al. 2000). Given the importance of the subhalo mass function as a clue to understanding of the galaxy luminosity function, however, what is also desired is the global subhalo mass function, i.e., the mass function of all the subhalos in the universe, irrespective of the host halos.

As for the analytic approach, the hindrance is the complexity of the subhalo evolution. For the mass function of dark halos, we already have a remarkably successful theory developed by Press & Schechter (1974, hereafter PS). The principle of the PS theory is this: the formation and evolution of dark halos can be traced by the linear theory, assuming (i) dark halos have no internal structure; (ii) dark halos form independently of their surroundings; (iii) dark halos do not lose mass in the evolution but only hierarchically merge via gravity. Unlike the case of the halo mass function, however, the subhalo mass function cannot be derived under such simple assumptions. The subhalos are, by definitions, the internal structures of the halos, being placed in highly dense surroundings, and thus the formation and evolution of the subhalos must depend strongly on their surroundings. Among the various consequences from the surrounding influences, the most significant one is the subhalo mass-loss: the subhalos do not only gravitationally merge but also get disrupted or at least lose considerable amount of their mass through the interaction with the surroundings. In fact, it has been demonstrated by several N-body simulations that the subhalos lose most of their mass throughout the evolution, contributing after all only $10 - 15\%$ of the total mass of the host halos (e.g., Tormen *et al.* 1998). There are three different processes that can drive the subhalos to lose mass: the global tides generated by the host halos, the dynamical frictions, and the close-encounters with the other subhalos. Apparently, the subhalo mass-loss is quite a complicated process, so that it would be practically impossible to take into account its effect fully in deriving the subhalo mass function analytically. That was why all previous analytic approaches had to make the unrealistic assumption that the subhalos do not lose mass during the evolution (Fujita et al. 2002; Sheth 2003; Blanton 2003).

However, the mass-loss phenomenon is the most essential feature of the subhalo evolution, which must be taken into account in order to estimate the subhalo mass function in any realistic sense. Here we attempt for the first time to estimate both the global and the local subhalo mass functions with the subhalo mass-loss effect taken into account. To make the theory analytically tractable, we still make some simplified assumptions that the subhalo mass-loss is mainly driven by the global tides, and that the condition for a subhalo

to survive the global tides is a simple function of the distance from its host halo.

2. FORMALISM

The (differential) global subhalo mass function, $\frac{dN(M_s)}{d\ln M_s}d\ln M_s$, is defined as the number density of subhalos in logarithmic mass range $[\ln M_s, \ln M_s + d\ln M_s]$. To estimate it, we assume the following.

(i) The gravitational collapse process to form dark matter halos follows the Top-hat spherical dynamics, according to which a dark halo of mass M forms at redshift z if the density contrast δ ($\delta \equiv \Delta\rho/\bar{\rho}$, $\bar{\rho}$: the mean mass density of the universe) of a Lagrangian region in the linear density field smoothed on mass scale of M satisfies the gravitational collapse condition of $\delta = \delta_c(z)$ where $\delta_c(z)$ is the critical density contrast at redshift z whose value depends on cosmology (Kitayama & Suto 1996). For a flat universe of closure density, $\delta_c(z) \approx 1.68(1+z)$ (Gunn & Gott 1972).

(ii) The mass function of dark halos is well evaluated by the PS theory as

$$\frac{dN(M)}{d\ln M} = \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M}\left|\frac{d\ln\sigma}{d\ln M}\right|\nu\exp\left(-\frac{\nu^2}{2}\right), \tag{1}$$

where $\nu \equiv \delta_c/\sigma(M)$ and $\sigma(M)$ is the rms density fluctuation of the linear density field on mass scale of M.

(iii) A dark halo hosts a multiple of subhalos, each of which rotates on a circular orbit. The global tidal field of the host halo strips the subhalos, which drives the subhalos to either get completely destroyed or survive but with reduced mass. A subhalo of initial mass M_1 before the tidal stripping effect at an initial distance r from its host halo of mass M_2 eventually survives the tidal stripping effect, ending up with having reduced mass of $M_s(< M_1)$, if M_1, M_2, and r, satisfy the condition

$$M_s = c_t\left(\frac{4\pi}{3}\bar{\rho}\right)\left(\frac{M_1}{M_2}\right)r^3, \tag{2}$$

where the proportionality constant c_t is a free parameter.

(iv) The spatial distribution of the subhalos inside their host halo follows that of the dark matter particles, i.e., the profile given by Navarro, Frenk, & White (1996) (hereafter, NFW):

$$P_{M_2}(r) \propto \frac{1}{(r/r_s)(1+r/r_s)^2} \tag{3}$$

where r_s is the scale radius.

It is worth noting that equation (2) is reminiscent of the familiar tidal-limit approximation, according to which a subhalo in the tidal field loses all mass beyond its tidal radius, r_t. We set c_t as a free parameter since its precise value depends on underlying assumptions: if the halo potentials can be approximated as point mass and the linear size of the subhalo is much smaller than the distance from the host halo, the tidal radius has a simple expression of $r_t = (c_t M_1/M_2)^{1/3} r$ with $c_t = 1/2$ (the Roche limit); if the effect of the centrifugal force is taken into account, then $c_t = 1/3$ (the Jacobi limit); if the halos are treated more realistically as the extended mass profiles, then the tidal radius has a more general expression (see, e.g., Tormen *et al.* 1998). Although equation (2) is an obvious oversimplification of real tidal mass-loss process (Hayashi et al. 2003), there are numerical clues that the fraction of the survival subhalos has a strong correlation with the distance from the host halos (Okamoto & Habe 1999). It implies that the tidal survival condition should depend on the subhalo orbital distance as well as the host halo and the subhalo mass. Therefore, equation (2) may be the simplest possible choice for the functional form of the tidal survival condition.

Let us first consider $P_{M_2}(M_1; r)$, the conditional probability that a subhalo has an initial mass greater than M_1 (before the tidal mass-loss) provided that it rotates upon a host halo of mass M_2 at a distance r. According to the hypothesis (i), $P_{M_2}(M_1; r) = P_{\delta_2=\delta_c}[\delta_1(r) \geq \delta_c]$ where δ_i (with $i = 1, 2$) is the density contrast of a Lagrangian region in the linear density field smoothed on mass scale M_i, and δ_{ci} is the critical value of δ_i. The conditional probability $P_{\delta_2=\delta_c}[\delta_1(r) \geq \delta_c]$ can be computed from the Gaussian probability density distribution with the help of the Bayes theorem. For the case of the sharp k-space filter, it has the following simple analytic form (Yano et al. 1996):

$$P_{M_2}(M_1; r) = \frac{1}{\sqrt{2\pi}} \int_{\beta}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx, \quad \text{with} \quad \beta \equiv \frac{1}{\sqrt{1-\gamma^2}} \left(\frac{\delta_{c1}}{\sigma_1}\right) \left[1 - \frac{\delta_{c2}}{\delta_{c1}} \frac{\sigma_c^2}{\sigma_2^2}\right]. \quad (4)$$

Here σ_i^2 (with $i = 1, 2$) and $\sigma_c^2(r)$ are the mass variance of the linear density field on mass scale of M_i, and the linear density cross correlation, respectively, given as

$$\sigma_i^2 = \int_{-\infty}^{\ln(k_{ci})} \Delta^2(k) d\ln k, \qquad \sigma_c^2 = \int_{-\infty}^{\ln(k_{c2})} \Delta^2(k) \frac{\sin kr}{kr} d\ln k \quad (5)$$

where Δ_k is the dimensionless power spectrum of the linear density field, $\gamma \equiv \sigma_c^2/(\sigma_1\sigma_2)$, and the integral upper limit k_{ci} is related to M_i by $k_{ci} = (6\pi^2\bar{\rho}/M_i)^{1/3}$. In fact, it was Yano et al. (1996) who first incorporated the spatial correlations between the dark halos themselves into the PS theory. Fujita et al. (2002) used the formalism of Yano et al. (1996) to estimate the local subhalo mass function with the spatial correlations between the host halos and the subhalos taken into account. However, Fujita et al. (2002) assumed that the spatialdistribution of the subhalos is uniform, and averaged $P_{M_2}(M_1; r)$ over r without taking into account the tidal mass-loss and its correlation with r.

Next, let us consider $P_{M_2}(r)$, the probability of finding a subhalo in a spherical shell of radius r and thickness dr around a host halo of mass M_2 in the Lagrangian space. According to the hypothesis (iv), it can be written as

$$P_{M_2}(r) = \frac{A}{(r/r_s)(1+r/r_s)^2} 4\pi r^2 dr, \quad \text{with} \quad A \equiv \frac{1}{4\pi r_s^3[\ln(1+\frac{R_2}{r_s}) - \frac{R_2}{r_s+R_2}]}. \tag{6}$$

where R_2 is the virial radius of M_2, and the amplitude A is determined to satisfy the normalization constraint of $\int_0^{R_2} P_{M_2}(r) = 1$. We approximate R_2 by the Top-hat radius of M_2, and adopt the empirical relation for r_s proposed by Klypin et al. (1999):

$$R_2 = \left[\frac{3M_2}{4\pi\bar{\rho}}\right]^{1/3}, \qquad r_s = \frac{R_2}{124}\left[\frac{M_2}{h^{-1}M_\odot}\right]^{0.084}. \tag{7}$$

It is also worth mentioning here that δ,R_2, r, and r_s are all measured in the Lagrangian space where the density field is still Gaussian.

The joint conditional distribution, $P_{M_2}(r, M_1)$, i.e., the probability of finding a subhalo of mass greater than M_1 at a distance r from a host halo of mass M_2, can be derived from equations (4) and (6) by using the Bayes theorem: $P_{M_2}(M_1, r) = P_{M_2}(M_1; r)P_{M_2}(r)$. The partial derivative, $\partial P_{M_2}(r, M_1)/\partial M_1$, is proportional to the fraction of the host halo volume occupied by those subhalos of mass M_1 at a distance r, $f_{M_2}(M_1, r)$, such that $f_{M_2}(M_1, r) = (M_2/\bar{\rho})|\partial P_{M_2}(r, M_1)/\partial M_1|$ where the proportionality factor $(M_2/\bar{\rho})$ is nothing but the average volume of the host halo. Following the familiar PS-like approach, the initial number density of the subhalos of mass M_1 at a distance r inside a host halo of mass M_2 equals $f_{M_2}(M_1, r)$ divided by the average volume of the subhalo $M_1/\bar{\rho}$ such that $dN_{M_2}(r, M_1)/(4\pi r^2 dr\ dM_1) = 2\,(\bar{\rho}/M_1)\, f_{M_2}(M_1)$ where the factor 2 is the normalization constant introduced by PS (see also, Peacock & Heavens 1990; Bond et al. 1991; Jedamzik 1995; Yano et al. 1996; Fujita et al. 2002). We end up with the following expression:

$$\frac{d^2N_{M_2}(r, M_1)}{4\pi r^2 dr\ d\ln\tilde{M}_1} = \sqrt{\frac{2}{\pi}}\left(\frac{1}{\tilde{M}_1}\right)\left|\frac{d\ln\sigma_1}{d\ln\tilde{M}_1}\right|\left(\frac{\xi}{1-\gamma^2}\right)\frac{A}{(r/r_s)(1+r/r_s)^2}\exp\left(-\frac{\xi^2}{2}\right). \tag{8}$$

where $\tilde{M}_1 \equiv M_1/M_2$ and $\xi \equiv \frac{\nu_1}{\sqrt{1-\gamma^2}}\left(1 - \frac{\nu_1}{\nu_2}\gamma\right)$ with $\nu_i \equiv \delta_{ci}/\sigma_i$ for $i = 1, 2$. Equation (8) is the *conditional* mass and spatial distribution of the subhalos provided that they are included in the host halos of mass M_2 before the tidal mass-loss. If the subhalo mass were conserved, the local subhalo mass function would be computed simply by integrating equation (8) over r. However, the subhalos in reality either get destroyed or lose mass, so that the subhalo mass function cannot be simply obtained in that way. Notwithstanding, the reduced mass of survived subhalos can be determined from the informations of M_1, M_2, and r according to the hypothesis (iii).

Hence, we find the local subhalo mass function after the tidal mass loss as

$$\frac{dN_{M_2}(M_s)}{d\ln M_s}d\ln M_s = \int_{r_c}^{R_2} 4\pi r^2 dr \int_{-\infty}^{0} d\ln \tilde{M}_1 \; \frac{dN_{M_2}(r, M_1)}{4\pi r^2 dr d\ln \tilde{M}_1} \; \delta_D\left(c_t \frac{4\pi}{3}\bar{\rho}\tilde{M}_1 r^3 - M_s\right). \quad (9)$$

where δ_D represents the Dirac-delta function, and r_c represents the lower limit for the subhalo survival: if a subhalo is located at a distance smaller than r_c, they get completely destroyed by the strong tidal stripping effect. The value of r_c has been empirically found to be a few times the scale radius r_s (Ghigna et al. 2000; Hayashi et al. 2003). The upper limit R_2 in the integration of r is set from the expectation that the subhalos should be inside the virial radius of the host halo. Note that equations (8) and (9) are all expressed in terms of the rescaled subhalo mass $\tilde{M}_1 = M_1/M_2$. Their dependence on the host halo mass comes only implicitly from γ and ξ. It implies that the subhalo mass function should not depend strongly on the host halo mass, consistent with numerical detections (e.g., De Lucia et al. 2003). Furthermore, we compute the contribution of the subhalos to the host halo mass by integrating equation (9) over M_s, and find that it is approximately 10% of the host halo mass, which is also consistent with numerical simulations (e.g., Tormen *et al.* 1998). Finally, the global subhalo mass function is obtained by multiplying the local subhalo mass function by the host halo mass function and integrating it over the host halo mass:

$$\frac{dN(M_s)}{d\ln M_s}d\ln M_s = \int_{-\infty}^{\infty} d\ln M_2 \; \frac{dN(M_2)}{d\ln M_2} \; \frac{dN_{M_2}(M_s)}{d\ln M_s}d\ln M_s, \quad (10)$$

where the host halo mass function $dN(M_2)/d\ln M_2$ is given in equation (1).

Figure 1 plots the cumulative local (upper panel) and the global (lower panel) mass functions at redshift $z = 0$ for the case of for a flat CDM cosmology (Bond & Efstathiou 1984) with the following choice of the cosmological parameters: the matter density $\Omega_m = 0.3$, the vacuum energy density $\Omega_\Lambda = 0.7$, the shape parameter $\Gamma = 0.5$, the dimensionless Hubble constant $h = 0.5$, and the rms density fluctuation on the scale of $8h^{-1}$Mpc, $\sigma_8 = 0.7$. For the local subhalo mass function, the host halo mass is chosen to be $M_2 = 10^{14} M_\odot h^{-1}$. In each panel, the solid and the dashed lines correspond to the cases of $c_t = 1/3$ and $c_t = 1/2$, respectively. For comparison, the subhalo mass function with no tidal effect is also plotted as dotted lines. It is clear from Figure 1 that both the global and the local cumulative subhalo mass functions are close to a power law $N(M_s) \sim M_s^l$, and that the tidal mass-loss effect changes the subhalo mass function in a self-similar manner. We find that the power-law slope of the local subhalo mass function is $l \sim -0.8$ while that of the global subhalo mass function is slightly sharper $l \sim -0.9$. The power-law slope of the global subhalo mass function gets sharper in the high mass section. It is due to that massive subhalos experience the stronger tidal stripping effect (eq. [2]) to lose more mass, and also that the dark halos which can afford to hosting such massive subhalos are rare whose number density decrease sharply (eq. [1]).

3. DISCUSSIONS AND CONCLUSIONS

We provided for the first time a theoretical formalism in which one can estimate analytically the global and the local mass distribution of dark matter subhalos that undergo tidal mass-loss process. Adopting the simple tidal-limit approximation, we showed that the resulting mass functions are consistent with what has been found in recent high-resolution N-body simulations, providing theoretical clues to the unique properties of the subhalo mass distribution: the power-law shape, weak dependence on host halo mass, self-similar change, and roughly 10% contribution of subhalos to the total mass.

Yet, it is worth mentioning that our subhalo mass functions are subject to several caveats. The most obvious one is that we have oversimplified the subhao mass-loss process, using the simple tidal limit approximation, and also ignored the effects of dynamical frictions and close encounters between subhalos. It has been shown by numerical simulations that the tidal limit approximation underestimates the subhalo mass-loss considerably (Hayashi et al. 2003). Although it was shown by simulations that the most dominant force that leads to the mass loss of the subhalos is the global tides (Okamoto & Habe 1999), the dynamical frictions and subhalo close-encounters may change the subhalo orbits making the subhalos more susceptible to the tidal forces (Tormen *et al.* 1998). We have also assumed simply that the subhalos rotate on stable circular orbits. However, in reality, the the subhalo orbits are quite eccentric, changing with time (Tormen *et al.* 1998; Hayashi et al. 2003). Definitely, it will be quite necessary to refine our formalism by making more realistic treatments of the subhalo evolution, especially its mass-loss process.

Finally, we conclude that our formalism is expected to provide an important first step toward realistic modeling of the abundance distribution of dark halo substructures.

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REFERENCES

Blanton, M. R. 2003, preprint (astro-ph/0304315)

Bond, J. R., & Efstathious, G. 1984, ApJ, 285, 45

Bond, J. R., Cole, S., Efstathious, G., & Kaiser, N. 1991, ApJ, 379, 440

De Lucia, G., Kauffmann, G., Springel, V., White, S. D. M. 2003, preprint (astro-ph/0306205)

Fujita, Y., Sarazin, C. L., Nagashima, M., & Yano, T. 2002, ApJ, 577, 11

Ghigna, S., Moore, B., Governato, F., Lake, G., Quinn, T., & Stadel, J. 2000, ApJ, 544, 616

Gunn, J. E., & Gott, J. R., 1972, ApJ, 176, 1

Hayashi, E., Navarro, J. F., Taylor, J. E., Stadel J., Quinn, T. 2002, ApJ, 584, 541

Jedamzik, K. 1995, ApJ, 448, 1

Kitayama, T., & Suto, Y., 1996, ApJ, 469, 480

Klypin, A., Kravtsov, A., Valenzuela, O., & Prada, F. 1999, ApJ, 522, 82

Moore, B., Calcaneo-Roldan, C., Stadel, J., Quinn, T., Lake, G., Ghigna, S., & Governato, F. 2001, Phys. Rev. D, 64, 063508

Navarro, J.F., Frenk, C.S., & White, S. D. M. 1996, ApJ, 462, 563

Okamoto, T., & Habe, A. 1999, ApJ, 516, 591

Peacock, J. A., & Heavens, A. F. 1990, MNRAS, 243, 133

Press, W., & Schechter, P., 1974, ApJ, 187, 425

Springel, V., White, S.D.M., Trmen, G., Kauffmann, G., 2001, MNRAS, 328, 726

Sheth, R. K. 2003, preprint

Sheth, R. K., & Jain, B. 2002, preprint (astro-ph/0208353)

Tormen, G., Diaferio, A., Syer, D. 1998, MNRAS, 299, 728

White, S. D. M., & Rees, M. J. 1978, ApJ, 183, 341

Yano, T., Nagashima, M., & Gouda, N. 1996, ApJ, 466, 1

Zhang, B., Wyse, R. F. G., Stiavelli, M., Silk, J. 2002, MNRAS, 332, 647

Zentner, A. R., & Bullock, J. S. 2003, preprint (astro-ph/0304292)

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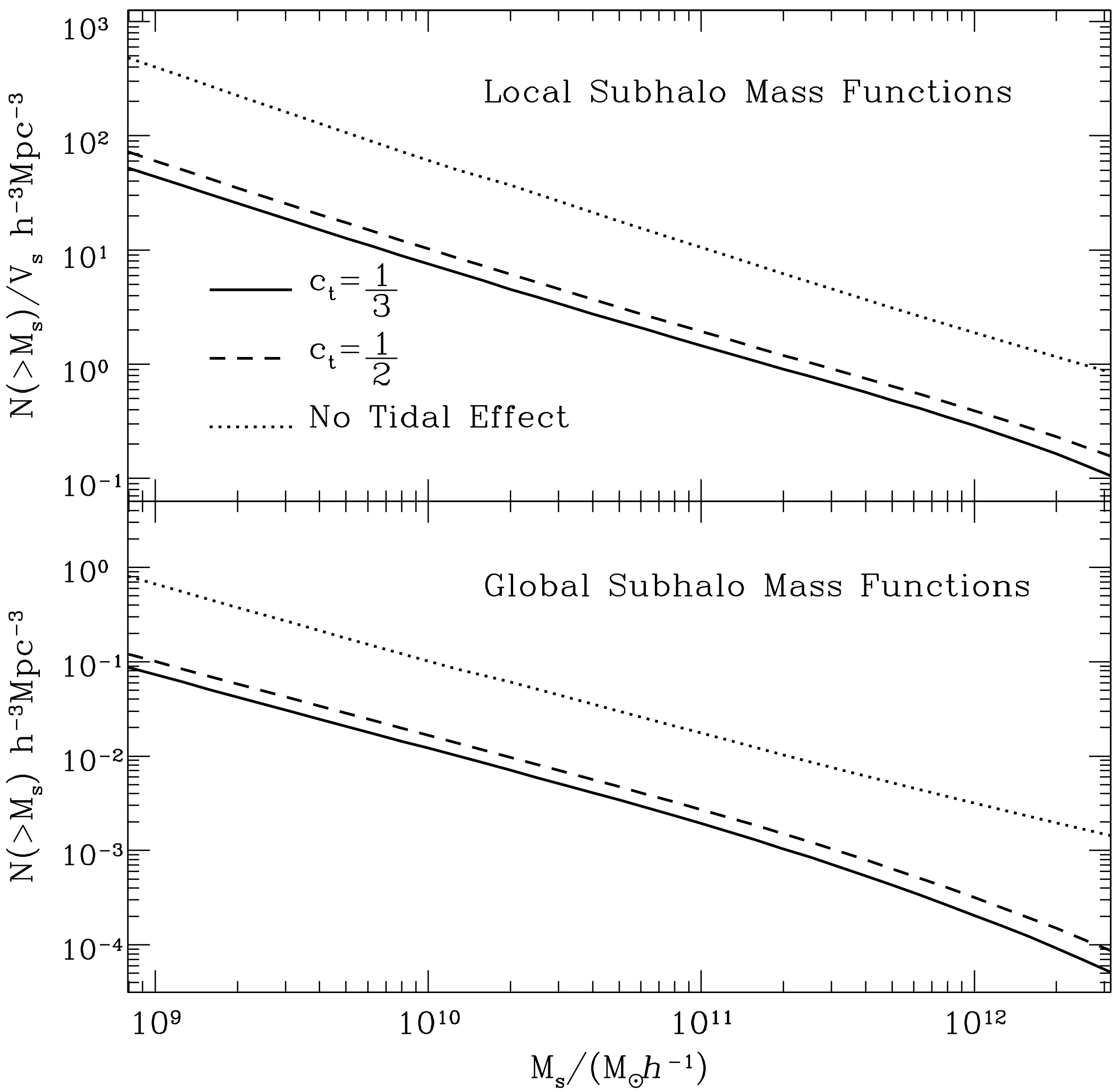

Fig. 1.— The mass functions of the dark halo substructures at redshift z=0: *Upper Panel*: the local distribution and *Lower Panel*: the global distribution. The solid and dashed lines correspond to the two different values of the free parameter c_t in the the tidal survival condition (see, eq. [2]). The dotted lines correspond to the case of no tidal stripping effect.